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Non-linear Programming for a η-approximation involving generalized Invexity

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Abstract: A framework is used to find optimal solution for a non-linear programming problem involving generalized invexity. Mon d-Weir duality results using η-approximation associated with primal objectives constituted using η-approximation. Further, various equivalence relations are obtaind. Duality results are proved.

1. Introduction: Convexity plays an important role in non-linear programming problem with single and multiple objectives. In the recent past, several classes of generalized convex functions and their analogous results were established by weakening convexity assumption. First, Hanson[11] introduced the concept of invexity and related results. In Ref [10], Craven named them as invex function. The initiative given by Hanson inspired many researchers to develop and establish further generalization of convexity.

The main purpose of this paper is to generate the different concepts of Antczak in [1], [2], [3], [4] to a case when the functions involved in objective as well as constraints are twice differentiable. For this purpose, we use Antczak's second order η-approximation method with respect to generalized invexity conditions. Further,

2. Fundamental Concepts and Definitions

The following definitions are used in the sequel.

Definition 2.1: If f: $X\rightarrow R$ is a differentiable function on a nonempty open set $X \subset R^n$, then f is said to be a ρ-invex (ρ-incave) at $x \in X$ on X if, for for all $x \in X$, we have,

$$f(x_o)- f(\mu_o) \geq \nabla f(\mu)\, \eta^T (x,\mu) + \rho\ \|\theta (x,\mu)\|^2 \qquad (1)$$

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$$f(x_o)- f(\mu_o) \leq \nabla f(\mu)\, \eta^T (x,\mu) + \rho \, \|\theta (x,\mu)\|^2 \qquad (2)$$

If the inequality (1) holds for any $x \rightarrow X$, then f is invex. Similarly, if the inequality (2) holds for any $x \in X$ then f is ρ-incave.

Definition 2.2: Let $X \subset R^n$ be an open nonempty set and let f: $X\rightarrow R$ be a differentiable function then, the function f is said to be ρ-pseudo-invex (ρ-pseudo-incave) at $x \in X$ on X if, for all $x \in X$, we have,

$$\nabla f(\mu)\, \eta^T (x,\mu) + \rho \, \|\theta (x,\mu)\|^2 \geq 0$$

$$\Rightarrow f(x) \geq f(\mu) \qquad (3)$$

$$\text{and, } \nabla f(\mu)\, \eta^T (x,\mu) + \rho \, \|\theta (x,\mu)\|^2 \leq 0$$

$$\Rightarrow f(x) \leq f(\mu) \qquad (4)$$

Definition 2.3: Let $X \subset R^n$ be an open nonempty set and let f: $X\rightarrow R$ be a differentiable function then, the function f is said to be ρ-quasi-invex (ρ-quasi-incave) at $\mu \in X$ on X if, for all $x \in X$, we have,

$$\nabla f(\mu)\, \eta^T (x,\mu) + \rho \, \|\theta (x,\mu)\|^2 > 0$$

$$\Rightarrow f(x) > f(\mu)$$

$$\text{and, } \nabla f(\mu)\, \eta^T (x,\mu) + \rho \, \|\theta (x,\mu)\|^2 < 0$$

$$\Rightarrow f(x) < f(\mu)$$

The following lemma is the consequence of the above definitions.

Lemma 2.1.

If f: $X\rightarrow R$ is a quasi- ρ-invex and pseudo- ρ-invex at at μ on X iff for all $x \in X$, we have,

$$f(x) \leq f(\mu) \Leftrightarrow \nabla f(\mu)\, \eta^T (x,\mu) + \rho \, \|\theta (x,\mu)\|^2 \leq 0$$

According to Refs ([ ],[ ]), we have

$$\rho\text{-invexity} \Rightarrow \text{pseudo- } \rho\text{-invexity} \Rightarrow \text{quasi- } \rho\text{-invexity}$$

(NLMP) Consider the following nonlinear constrained mathematical programming problem

Min $f(x) = (f_1(x), f_2(x), f_3(x), \ldots\ldots, f_p(x))$

Subject to $g_j(x) \leq 0$, j=1,2,3,……, m; where f_i: $X \rightarrow R$ and $g= (g_1, g_2, g_3, \ldots\ldots g_m)$ are differentiable functions on an open set $X \subset R^n$. Let us denote the set of feasible solution in (NLMP) as

$$F = \{x \leftarrow X: g_j(x) \leq 0, j=1,2,3,\ldots..,m\}$$

Definition 2.5: Any point $\bar{x} \rightarrow X$ is known as an optimal point in (NLMP) if for all $x \leftarrow F$, we have,

$$f(x) \geq f(\bar{x})$$

Note: According to Bazara et al [5], the Karush – Kuhn – Tucker condtion are necessary optimality for such optimization problems. But the converse may not be true.

Theorem 2.6: According to Bazara et al [5], if $\bar{x}$ is an optimal solution in (NLMP), at which some constraint qualification C[12] satisfies, then $\exists\ \bar{\xi} \in R_+^m$, and $\bar{\xi} \geq 0$, such that,

$$\nabla f(x) + \bar{\xi}\, \nabla g(\bar{x}) = 0$$

$$\bar{\xi}\, g(\bar{x}) = 0$$

$$\xi \geq 0$$

3. Optimality Criteria using η-approximation

Suppose $\bar{x}$ is a feasible solution in (NLMP). Now, we considered Antczak [1] η-approximated optimization problem $(P_\eta(\bar{x}))$ as

$$\text{Min } f(\bar{x}) + \nabla f(\bar{x})\, \eta^T (x, \bar{x}) \text{ subject to}$$

$$g_j(\bar{x}) + \nabla g_j(\bar{x})\, \eta^T (x, \bar{x}) \leq 0; \quad j=1,2,3,\ldots\ldots\ldots,m$$

where f , g and x have their usual meanings.

Let us denote the set of feasible solutions for $(P_\eta (\bar{x}))$ as

$$F(\bar{x}) = \{ \{x \rightarrow X: g_j(\bar{x}) + \nabla g\ (\bar{x})\, \eta^T (x, \bar{x}) \leq 0, j=1,2,3,\ldots..,m\}$$

To state and prove optimality criteria and duality results, we need some extra conditions, which are imposed on the function η.

Condition (C): According to Antczak, the condition may be defined as: Let us denote by $\eta(.,,\bar{x})$ the function X$\rightarrow \eta$ (x, $\bar{x}$). It is known that η satisfies the condition at a point $\bar{x}$. Further, when η (.,,$\bar{x}$) is a differentiable function at the point $x=\bar{x}$, with respect to the first component and satisfies the conditions $\eta(\bar{x},\bar{x})=0$ and $\eta_x(\bar{x},\bar{x})=\alpha.1$. Here α is any real number and η_x is a first order derivative.

Now Karush – Kuhn – Tucker optimality conditions (necessary) for the problem $P_n(\bar{x})$ have the following form:

Theorem 3.1: If $\bar{x}$ is an optimal solution in $(P_n(\bar{x}))$ at which a constrained constraint (ref [12]) is satisfied and we assume that η also satisfies the condition (C) then $\exists\ \bar{\xi} \in R^m_+$, and $\bar{\xi} \geq 0$, such that,

$$\nabla f(\bar{x}) + \bar{\xi}\, \nabla g(\bar{x}) = 0$$

$$\bar{\xi}\, g(\bar{x}) = 0$$

$$g \geq 0$$

Proof: Similar as in ref [1].

Note: From practical point of view, some extra care has been taken on the condition $\eta_x(x,\bar{x})=\alpha$ in the given condition (C). Without loss of generality, we can impose an extra condition, which was used earlier in ref [1]. By assuming this, the optimality condition for (NLMP) at $P_n(\bar{x})$ are the same.

Theorem 3.2: If g_j, j=1,2,3,……, m be a set of active constraint function at $\bar{x} \in$ F. However, we assume that the set of active constraints are quasi- ρ-invex and pseudo- ρ-incave at $\bar{x}$ on F, and then the set of feasible solutions for both the problems are same.

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Proof: Similar to problem (NLMP) and $P_n(\bar{x})$. The following theorem shows the equivalence between (NLMP) and $P_n(\bar{x})$.

Theorem 3.3: If $\bar{x}$ is an optimal solution in (NLMP) and satisfies a suitable constraint qualification [12]at $\bar{x}$. If η satisfies condition (C) then $\bar{x}$ is also an optimal solution in $(P_n(\bar{x}))$.

Proof: we prove this by contradiction. We assume that $\bar{x}$ is optimal and satisfies a constraint qualification. Then, $\exists\ \bar{\xi} \geq 0$, such that the KKC are satisfied. Suppose $\bar{x}$ is not optimal in $(P_n(\bar{x}))$ which mean $\exists\ \bar{x}$ feasible for $(P_n(\bar{x}))$ such that

$$f(\bar{x}) + \nabla f(\bar{x})\ \eta^T (x, \bar{x}) < f(\bar{x}) + \nabla f(\bar{x})\ \eta^T (x, \bar{x}) + \rho\|\theta (x, \bar{x})\| + \rho\|\theta (x, \bar{x})\|^2$$

$$\text{and so } \nabla f(\bar{x})\ \eta^T (x, \bar{x}) + \rho\|\theta (x, \bar{x})\|^2 < 0$$

Since $\bar{\xi} \geq 0$ and the feasibility of $\bar{x}$ is in $(P_n(\bar{x}))$,

$$\text{we get } \bar{\xi}\ \bar{g}\ (\bar{x}) + \bar{\xi}\ \ \nabla g_j(\bar{x})\ \eta^T (x, \bar{x}) + \rho\|\theta (x, \bar{x})\|^2 \leq 0$$

But, from known Karussh Kauhn Tucker, we obtain

$$[\nabla f(\bar{x}) + \bar{\xi}\ \ \nabla g_j(\bar{x})\ \eta^T (\bar{x}, \bar{x}) + \rho\|\theta (\bar{x}, \bar{x})\|^2 < 0]$$

which is a contradiction to our assumption. Then $\bar{x}$ is an optimal in $(P_n(\bar{x}))$.

We prove the following theorem by making use of ρ-invexity on both objective as well as constraint functions with the help of condition (C).

Theorem 3.4: Suppose $\bar{x}$ is known as an optimal solution in problem $(P_n(\bar{x}))$ and satisfies some constraint qualification (Ref [12]) at $\bar{x}$. Moreover, the function f and g are ρ-invex at $\bar{x}$ on F with respect to same η and θ, and η satisfies condition (C), then $\bar{x}$ is optimal in (P).

Proof: We prove this result by using contradiction.

Suppose $\bar{x}$ is optimal in NLMP $(P_n(\bar{x}))$, hence, the inequality

$$f(x) + \nabla f(\tilde{x})\, \eta^T (x, \tilde{x}) + \rho\|\theta (x, \tilde{x})\|^2 \geq f(\tilde{x}) + \nabla f(\tilde{x})\, \eta^T (x, \tilde{x}) + \rho\|\theta (x, \tilde{x})\|^2 \quad (5)$$

holds for all $x \in F$.

By 5, we have

$$\nabla f(\bar{x})\, \eta^T (x, \bar{x}) + \rho\|\theta (x, \bar{x})\|^2 \geq 0 \quad (6)$$

Holds for all $x \in F$.

Next, let us assume g is ρ-invex at $\bar{x}$ on F. Then it easily follows that $F \subset F(\bar{x})$. Let us suppose that $\bar{x}$ is not optimal solution in (NLMP). Then, $\exists\, \tilde{x}$, which is also feasible for (NLMP) such that

$$f(\tilde{x}) < f(\bar{x}). \quad (7)$$

By $\tilde{x} \in F$ and from $F \subset F(\tilde{x})$, it follows that $\tilde{x}$ is also feasible in $(P_n(\bar{x}))$. By assumption, f is ρ-invex at $\bar{x}$ on F, by using EQ.7, we obtain

$\nabla f(\bar{x})\, \eta^T (\tilde{x}, \bar{x}) + \rho\|\theta (\tilde{x}, \bar{x})\|^2 < 0$ which is a contradiction to EQ. 6. Hence $\bar{x}$ is optimal in (NLMP).

4. Duality

In this section we generalize Mond – Weir type duality [11] of the (NLMP) which associated with η-approximation of mathematical programming problem.

Consider

(NMWD) Max F(y)

Subject to $\nabla f(y) + \xi^T \nabla g(y) = 0$

$\xi_j^T g_j(y) \geq 0$, j=1,2,3……,m

$x, \xi \geq 0$.

Let us denote the set of feasible solutions as

$$M= \{(Y, \xi) \in X \times R_+^m : \nabla f(y) + \xi^T \nabla g_j(y) = 0,\ \xi_j\, g_j(y) \geq 0,\ j=1,2,3,….,m\}$$

Also, $Y=\{y \in X: (y, \xi) \in M\}$

$NMED_{\eta}$ $(\bar{x})$:

$$\text{Max } f(\bar{x}) + \nabla F(\bar{x})\, \eta^{T} (y, \bar{x})$$

$$\text{Subject to } \nabla F(\bar{x}) + \xi \nabla G(\bar{x}) = 0$$

$$\xi_{j} (G_{j} (\bar{x}) + \nabla G_{j}(\bar{x})\, \eta^{T} (y,\bar{x}) \geq 0 \quad y \in x,\ \xi \geq 0.$$

Theorem 4.1: Strong Duality

If $\bar{x}$ is an optimal point in (NLMP) and satisfies a constraint qualification ([12]) at $\bar{x}$, Also, f and g are ρ-invex at $\bar{x}$ on x with respect to η and θ, Then ∃ $\bar{\xi} > 0$ such that $(\bar{x}, \bar{\xi})$ is optimal in (NMWD).

Proof: Proof is similar.

Theorem 4.2: Converse Duality

If $(\bar{y}, \bar{\xi})$ is an optimal solution in (NMWD) such that $g(\bar{y}) = 0$. Further, we assume that the function f and g are ρ-invex at $\bar{y}$ on X w.r.t some η and θ. Then $\bar{y}$ is an optimal solution in (NLMP).

Proof: We prove this by contradiction.

Now, we prove that $(\bar{y}, \bar{\xi})$ is also optimal in $(NMWD_{\eta}$ $(\bar{y}))$.

We assume that $(\bar{y}, \bar{\xi})$ is not optimal Then, ∃ (y,ξ) in $(NMWD_{\eta}$ $(\bar{y}))$ such that

$$f(y) + \nabla f(\bar{y})\, \eta^{T} (y, \bar{y}) + \rho\|\theta (y, \bar{y})\|^{2} < f(y) + \nabla f(\bar{y})\, \eta^{T} (y, \bar{y}) + \rho\|\theta (y, \bar{y})\|^{2} \tag{8}$$

$$\Rightarrow \nabla f(\bar{y})\, \eta^{T} (y, \bar{y}) + \rho\|\theta (y, \bar{y})\|^{2} > 0 \tag{9}$$

By assumption, we have $g(\bar{y}) = 0$.

Now, EQ. 9 gives

$$\xi\,(g(\bar{y}) + \nabla g(\bar{y})\,\eta^T (y, \bar{y}) + \rho\|\theta\,(y, \bar{y})\|^2 < 0$$

This is a contradiction to our assumption.

Again, we have to prove that $\bar{y}$ is also an optimal solution in (NLMP($\bar{y}$)).

For this, consider an optimization problem of the form:

(NLMP($\bar{y}$)): Min $f(\bar{y}) + \nabla f(\bar{y})\,\eta^T (x, \bar{y}) + \rho\|\theta\,(x, \bar{y})\|^2$

Subject to $g_j(\bar{y}) + \nabla g_j(\bar{y})\,\eta^T (x, \bar{y}) + \rho\|\theta\,(x, \bar{y})\|^2 = 0,\;\; x \in X$

Proof: The proof is based on contradiction.

Let us assume that $\bar{y}$ is not optimal in (NLMP($\bar{y}$)). Then $\exists$ x, which is feasible in (NLMP($\bar{y}$)), such that,

$$f(\bar{y}) + \nabla f(\bar{y})\,\eta^T (x, \bar{y}) + \rho\|\theta\,(x, \bar{y})\|^2 < f(\bar{y}) + \nabla f(\bar{y})\,\eta^T (\bar{y}, \bar{y}) + \rho\|\theta\,(x, \bar{y})\|^2$$

$$\Rightarrow \nabla f(\bar{y})\,\eta^T (x, \bar{y}) + \rho\|\theta\,(x, \bar{y})\|^2 < 0$$

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